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ON SYMMETRIC FUNCTIONS.

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[Continued from January Number.]

This term is ever found in

$$\begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix} p_{i_1}p_{i_2}p_{i_3}$$

where p_{i_1} is to be applied to the j_1st line, p_{i_2} to the j_2d line, and p_{i_3} to the j_3d line. The term then corresponds to the substitution $\begin{pmatrix} j_1 & j_2 & j_3 \\ 1 & 2 & 3 \end{pmatrix}$, and has the same sign as before. We are now ready to state the general case.

g. We must show that it is indifferent whether we apply the exponents in all possible permutations to the columns or to the rows of D, *i. e.*, that to every term of the second formation corresponds the same term in the first formation, and then since the second formation is the straightforward product of D and Σ, it will follow that the first formation is equal to the product of D and Σ. We may denote the second and first formations by

$$\begin{vmatrix} \alpha_1^0 & \alpha_1 & \alpha_1^2 \ldots\ldots \alpha_1^{n-1} \\ \alpha_2^0 & \alpha_2 & \alpha_2^2 \ldots\ldots \alpha_2^{n-1} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ \alpha_n^0 & \alpha_n & \alpha_n^2 \ldots\ldots \alpha_n^{n-1} \end{vmatrix} \begin{matrix} p_{i_1} \\ p_{i_2} \\ \vdots \\ p_{i_n} \end{matrix} \text{ and } \begin{vmatrix} \alpha_1^0 & \alpha_2^0 & \alpha_3^0 \ldots\ldots \alpha_n^0 \\ \alpha_1 & \alpha_2 & \alpha_3 \ldots\ldots \alpha_n \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \alpha_3^{n-1} \ldots \alpha_n \end{vmatrix} p_{i_1}p_{i_2}\cdot\cdot p_{i_n}.$$

$i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers 1, 2, 3, ... n, and the expression on the left signifies that D has been multiplied by $\alpha_1{}^{p_{i_1}}\alpha_2{}^{p_{i_2}}\alpha_n{}^{p_{i_n}}$ of $\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$, the first row by $\alpha_1{}^{p_{i_1}}$, the second by $\alpha_2{}^{p_{i_2}}$, ... the nth by $\alpha_n{}^{p_{i_n}}$, or more briefly that p_{i_1} is to be applied as exponent to the elements of the first row, p_{i_2} in the same way to those of the second, ... and p_{i_n} similarly to those of the nth row, while the expression on the right (D with columns changed into rows and horizontal line of p's) shall signify that the p's are to be applied arbitrarily as exponents to the lines of D as written, the same thing as applying them to the columns as before written, and in such order that p_{i_1} is applied to the j_1st line, p_{i_2} to the j_2d line, ...p_{i_n} to the j_nth line.

h. Any term $\alpha_1{}^{p_{i_1}+j_1-1}\alpha_2{}^{p_{i_2}+j_2-1}\alpha_3{}^{p_{i_3}+j_3-1}\ldots\ldots\alpha_n{}^{p_{i_n}+j_n-1}$ (an expression which is seen to be an $(n\,!)^2$ valued function when one permutes the n p's in all possible ways, and also the n j's) correspond, with reference to the expression on the left hand, to the $(n\,!)^2$ valued substitution

$$s_1 s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix}$$

the first factor referring to the vertical series of p's, the second afterwards to the determinant, when the indicated multiplications have been performed. With reference to the right hand expression, corresponds the $(n\,!)^2$ valued substitution

$$s_1{}^1 s_2{}^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ i_1 & i_2 & i_3 & \ldots\ldots & i_n \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix},$$

in a similar manner, the first factor referring to the horizontal series of p's, and the second to the determinant after the p's have been applied to the elements, to the same term. It is clear that, numerically at least, the term is the same, for we must apply p_{i_1} to the j_1st line, and then take the first column ; this gives $\alpha_1{}^{p_{i_1}+j_1-1}$; next we must apply p_{i_2} to the j_2d line, and take the second column ; this gives $\alpha_2{}^{p_{i_2}+j_2-1}$; in general we must apply p_{i_r} to the j_rth line, and then take the rth column ; this gives $\alpha_r{}^{p_{i_r}+j_r-1}$, and the product $\prod\limits_{r=1}^{r=n}\alpha_r{}^{p_{i_r}+j_r-1}$ is numerically the term in question. The first substitutions s_1 and $s_1{}^1$ in either case have no effect on the sign of the term ; they merely assign the p's to their proper lines. The second substitutions

$$s_2 = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ j_1 & j_2 & j_3 & \ldots\ldots & j_n \end{pmatrix} \text{ and } s_2{}^1 = \begin{pmatrix} j_1 & j_2 & j_3 & \ldots\ldots & j_n \\ 1 & 2 & 3 & \ldots\ldots & n \end{pmatrix}$$

are reciprocals and have the same modulus or sign factor.

i. We have proved : The compound substitution $s_1 s_2$ corresponds to all the $(n\,!)^2$ terms which $D\Sigma\alpha_1{}^{p_1}\alpha_2{}^{p_2}\ldots\ldots\alpha_n{}^{p_n}$ is capable of having. Similarly the compound substitution $s_1{}^1 s_2{}^1$ which is simultaneous with $s_1 s_2$ and depends thereon corresponds to the $(n\,!)^2$ terms of $(p_1 p_2 \ldots\ldots p_n)$ and the terms corres-

ponding to $s_1^1 s_2^1$ are identical term by term with the terms corresponding to $s_1 s_2$. Therefore the theorem follows that

$$D\Sigma\alpha_1^{p_1}\alpha_2^{p_2}\ldots\ldots\alpha_n^{p_n}=(p_1 p_2\ldots\ldots p_n)=\Sigma(p_{i_1}\ \ p_{i_2}+1\ \ p_{i_3}+2\ldots\ldots p_{i_n}+n-1),$$

where $i_1, i_2, \ldots\ldots i_n$ form a permutation of the numbers $1, 2, \ldots\ldots n$.

B. Another Method. Elimination by Means of Symmetric Functions.

The problem of eliminating the variable between two binary forms by means of symmetric functions requires the calculation of the latter, and thus leads to the demand for symmetric functions as a whole. The calculation of all eliminants or resultants in succession is therefore, from this standpoint, the systematic calculation of all symmetric functions. In other words, the problems of calculating all resultants and of calculating all symmetric functions are identical. Given all resultants, we may write down the values of all symmetric functions. Given all symmetric functions, we may write down the values of all resultants. This idea is fruitful in giving rise to the following method of solving both problems simultaneously, and in yielding symmetric functions as a whole. The method will be illustrated and explained by one or two earlier cases, from which it will be seen that it can be carried as far as one pleases.

1. Two Quadratic Forms.

(1). *The Resultant.*

The resultant of two forms $a_0x^2+a_1x+a_2$ and $b_0x^2+b_1x+b_2$ of the second degree is (cf. p. 4)

$$b_0^2(a_0\beta_1^2+a_1\beta_1+a_2)(a_0\beta_2^2+a_1\beta_2+a_2)=$$

$$a_0^2(b_0\alpha_1^2+b_1\alpha_1+b_2)(b_0\alpha_2^2+b_1\alpha_2+b_2)=$$

$$b_0^2(a_0^2\Sigma\beta_1^2\beta_2^2+a_0a_1\Sigma\beta_1^2\beta_2+a_0a_2\Sigma\beta_1^2+a_1^2\Sigma\beta_1\beta_2+a_1a_2\Sigma\beta_1+a_2^2)=$$

$$a_0^2(b_0^2\Sigma\alpha_1^2\alpha_2^2+b_0b_1\Sigma\alpha_1^2\alpha_2+b_0b_2\Sigma\alpha_1^2+b_1^2\Sigma\alpha_1\alpha_2+b_1b_2\Sigma\alpha_1+b_2^2).$$

(2). *Aronhold's Operator.*

Applying Aronhold's Operator $\delta=b_0D_{a_0}+b_1D_{a_1}+b_2D_{a_2}$ on the first form of the resultant, first, using $b_0D_{a_0}$, then $b_1D_{a_1}$, and then $b_2D_{a_2}$, and denoting the coefficient of a_ia_k in the resultant by $|\ a_ia_k\ |=b_0^2\Sigma\beta_1^{2-i}\beta_2^{2-k}$, we get,

$$2a_0b_0\ |\ a_0^2\ |\ +b_0a_1\ |\ a_0a_1\ |\ +b_0a_2\ |\ a_0a_2\ |$$

$$+a_0b_1\ |\ a_0a_1\ |\ +2a_1b_1\ |\ a_1^2\ |\ +b_1a_2\ |\ a_1a_2\ |$$

$$+a_0b_2\ |\ a_0a_2\ |\ +a_1b_2\ |\ a_1a_2\ |\ +2a_2b_2\ |\ a_2^2\ |\equiv 0.$$

(3). *Identical Relations between Symmetric Functions.*

Since the expressions within the vertical strokes are functions of the b's,

and we can factor by columns, and take out the factors a_0, a_1, a_2 and since, for the rest, the whole expression is zero, whatever the values of the a's, it follows that their coefficients are zero, and identically zero, for they are zero for all values of the b's. We thus get :

$$2b_0 \mid a_0^2 \mid + b_1 \mid a_0 a_1 \mid + b_2 \mid a_0 a_2 \mid \equiv 0, \text{ coefficient of } a_0,$$

$$b_0 \mid a_0 a_1 \mid + 2b_1 \mid a_1^2 \mid + b_2 \mid a_1 a_2 \mid \equiv 0, \text{ coefficient of } a_1,$$

$$b_0 \mid a_0 a_2 \mid + b_1 \mid a_1 a_2 \mid + 2b_2 \mid a_2^2 \mid \equiv 0, \text{ coefficient of } a_2.$$

From these identities also follows :

$$\begin{vmatrix} 2 \mid a_0^2 \mid & \mid a_0 a_1 \mid & \mid a_0 a_2 \mid \\ \mid a_1 a_0 \mid & 2 \mid a_1^2 \mid & \mid a_1 a_2 \mid \\ \mid a_2 a_0 \mid & \mid a_2 a_1 \mid & 2 \mid a_2^2 \mid \end{vmatrix} \equiv 0,$$

an identical relation between the six symmetric functions of a quadratic form which enter into the resultant, with another quadratic form. It is seen to be a symmetric determinant.

(4). *Application of Relations to Calculate Symmetric Functions.*

We may use the preceding identities to find the symmetric functions involved, of which it may be taken for granted that we know

$$\mid a_0^2 \mid, \mid a_1^2 \mid, \mid a_2^2 \mid, \text{ and } \mid a_1 a_2 \mid, \text{ or } b_0^2 \Sigma(\beta_1\beta_2)^2, b_0^2 \Sigma\beta_1\beta_2, b_0^2 \Sigma(\beta_1\beta_2)^0,$$

$$\text{and } b_0^2 \Sigma\beta_1 \text{ equal to } b_2^2,\ b_0 b_2,\ b_0^2, \text{ and } -b_0 b_1.$$

Using these values with the first two identities, we have :

$$b_1 \mid a_0 a_1 \mid + b_2 \mid a_0 a_2 \mid = -2b_0 b_2^2$$

$$b_0 \mid a_0 a_1 \mid + 0 \mid a_0 a_2 \mid = -b_0 b_1 b_2.$$

Of these the second gives $\mid a_0 a_1 \mid = -b_1 b_2$, and by substituting this value in the first, $\mid a_0 a_2 \mid = -2b_0 b_2 + b_1^2$, the same results as appear in the table on page 5, when fx is changed into ϕx.

2. Two Cubic Forms.

We will next obtain relations between the symmetric functions which occur in the resultant of two forms of the third degree.

(1). *The Resultant.*

The resultant is equal to

$$a_0^3(b_0\alpha_1^3 + b_1\alpha_1^2 + b_2\alpha_1 + b_3)(b_0\alpha_2^3 + b_1\alpha_2^2 + b_2\alpha_2 + b_3)$$

$$\times (b_0\alpha_3^3 + b_1\alpha_3^2 + b_2\alpha_3 + b_3) = (-1)^{3\times 3}$$

$$b_0^3(a_0\beta_1^3+a_1\beta_1^2+a_2\beta_1+a_3)(a_0\beta_2^3+a_1\beta_2^2+a_2\beta_2+a_3)(a_0\beta_3^3+a_1\beta_3^2+a_2\beta_3+a_3)$$

$$=-(a_0^3\mid 0^3\mid+a_0^2a_1\mid 0^21\mid+a_0^2a_2\mid 0^22\mid+a_0^2a_3\mid 0^23\mid+a_0a_1^2\mid 01^2\mid$$

$$+a_0a_2^2\mid 02^2+a_0a_3^2\mid 03^2\mid+a_1^3\mid 1^3\mid+a_1^2a_2\mid 1^22\mid+a_1^2a_3\mid 1^23\mid$$

$$+a_1a_2^2\mid 12^2\mid+a_1a_3^2\mid 13^2\mid+a_2^3\mid 2^3\mid+a_2^2a_3\mid 2^23\mid+a_2a_3^2\mid 23^2\mid+a_3^3\mid$$

$$3^3\mid+a_0a_1a_2\mid 012\mid+a_0a_1a_3\mid 013\mid+a_0a_2a_3\mid 023\mid+a_1a_2a_3\mid 123\mid)^*.$$

(2). *Aronhold's Operator and the Identical Relations.*

Applying Aronhold's operators and collecting the coefficients of

$$a_0^2,\ a_1^2,\ a_2^2,\ a_3^2,\ a_0a_1,\ a_0a_2,\ a_0a_3,\ a_1a_2,\ a_1a_3,\ a_2a_3,$$

we have, putting each equal to zero :

$3b_0\mid 0^3\mid+b_1\mid 0^21\mid+b_2\mid 0^22\mid+b_3\mid 0^23\mid\equiv 0$, coefficient of a_0^2,

$b_0\mid 01^2\mid+3b_1\mid 1^3\mid+b_2\mid 1^22\mid+b_3\mid 1^23\mid\equiv 0$, coefficient of a_1^2,

$b_0\mid 02^2\mid+b_1\mid 12^2\mid+3b_2\mid 2^3\mid+b_3\mid 2^23\mid\equiv 0$, coefficient of a_2^2,

$b_0\mid 03^2\mid+b_1\mid 13^2\mid+b_2\mid 23^2\mid+3b_3\mid 3^3\mid\equiv 0$, coefficient of a_3^2,

$2b_0\mid 0^21\mid+2b_1\mid 01^2\mid+b_2\mid 012\mid+b_3\mid 013\mid\equiv 0$, coefficient of a_0a_1,

$2b_0\mid 0^22\mid+b_1\mid 012\mid+2b_2\mid 02^2\mid+b_3\mid 023\mid\equiv 0$, coefficient of a_0a_2,

$2b_0\mid 0^23\mid+b_1\mid 013\mid+b_2\mid 023\mid+2b_3\mid 03^2\mid\equiv 0$, coefficient of a_0a_3,

$b_0\mid 012\mid+2b_1\mid 1^22\mid+2b_2\mid 12^2\mid+b_3\mid 123\mid\equiv 0$, coefficient of a_1a_2,

$b_0\mid 013\mid+2b_1\mid 1^23\mid+b_2\mid 123\mid+2b_3\mid 13^2\mid\equiv 0$, coefficient of a_1a_3,

$b_0\mid 023\mid+b_1\mid 123\mid 2b_2\mid 2^23\mid+2b_3\mid 23^2\mid\equiv 0$, coefficient of a_2a_3.

(3). *Analysis of the Operator into Three Operators.*

We may notice the formation of these equations. They contain three operators ; an operator 0, 1, 2, 3 applied to the indices of the a's whose coefficient we seek, gives the combinations within the strokes ; b_0, b_1, b_2, b_3 give the literal coefficients and when the exponent of one of the indices in the strokes exceeds the exponent of the same index of the a's, whose coefficient we seek, this exponent becomes the numerical coefficient of the term in question. *E. g.*, for the coefficient of a_2^2, we apply 0, 1, 2, 3, to $\mid 2^2\mid$ and get

*The coefficient of $a_ia_ka_j$ has been farther abbreviated to $\mid i\ k\ j\mid$ for obvious reasons.

$| 02^2 |$, $| 12^2 |$, $| 2^3 |$, $| 2^2 3 |$ since here $| i\ k | = | k\ i |$.
b_0, b_1, b_2, b_3, are the corresponding literal coefficients ;
1, 1, 3, 1 are the corresponding numerical coefficients, and

$b_0 | 02^2 | + b_1 | 12^2 | + 3b_2 | 23^3 | + b_3 | 2^2 3 |$ is the coefficient of a_2^2. Again, for the coefficient of $a_1 a_3$, 0, 1, 2, 3, to $| 13 |$ give

$| 013 |$, $| 1^2 3 |$, $| 123 |$, $| 13^2 |$ also we have
b_0, b_1, b_2, b_3, and
1, 2, 1, 2 ; finally

$b_0 | 013 | + 2b_1 | 1^2 3 | + b_2 | 123 | + 2b_3 | 13^2 | =$ the coefficient of $a_1 a_3$.

(4). *Determinant Relations between Symmetric Functions.*

From the first four equations of (2) we have by elimination of b_0, b_1, b_2, b_3

$$\begin{vmatrix} 3\,|\,0^3\,| & |\,0^2 1\,| & |\,0^2 2\,| & |\,0^2 3\,| \\ |\,01^2\,| & 3\,|\,1^3\,| & |\,1^2 2\,| & |\,1^2 3\,| \\ |\,02^2\,| & |\,12^2\,| & 3\,|\,2^3\,| & |\,2^2 3\,| \\ |\,03^2\,| & |\,13^2\,| & |\,23^3\,| & 3\,|\,3^3\,| \end{vmatrix} \equiv 0.$$

In a similar way, using four equations at a time of the ten, we should have $\frac{10.9.8.7}{1.2.3.4} = 210$ identically vanishing determinant relations between the symmetric functions involved.

(5). *The Calculation of the Functions by the Identical Equations.*

a. Of the stroked elements in the equations of (2) we know $| 0^3 |$, $| 1^3 |$, $| 2^3 |$, $| 3^3 |$, $| 2^2 3 |$, $| 23^2 |$. They are $b_0^3 \Sigma(\beta_1 \beta_2 \beta_3)^3$, $b_0^3 \Sigma(\beta_1 \beta_2 \beta_3)^2$, $b_0^3 \Sigma(\beta_1 \beta_2 \beta_3)$, $b_0^3 \Sigma(\beta_1 \beta_2 \beta_3)^0$, $b_0^3 \Sigma\beta_1 \beta_2$, $b_0^3 \Sigma\beta_1$, equal to $-b_3^3$, $b_0 b_3^2$, $-b_0^2 b_3$, b_0^3, $b_0^2 b_2$, $-b_0^2 b_1$, respectively.

b. We may use them for solving the functions as follows : The equations of (2) become in order,

$$3b_0^4 \Sigma(\beta_1\beta_2\beta_3)^3 + b_1 b_0^2 \Sigma(\beta_1\beta_2\beta_3)^2 b_0 \Sigma\beta_1\beta_2 + b_2 b_0 \Sigma\beta_1\beta_2\beta_3 b_0^2 \Sigma\beta_1^2\beta_2^2$$

$$+ b_0^3 b_3 \Sigma\beta_1^3\beta_2^3 = 0, \text{ or } -3b_0 b_3^3 + b_1 b_2 b_3^2 - b_2^2 b_0^2 \Sigma\beta_1^2\beta_2^2 + b_3 b_0^3 \Sigma\beta_1^3\beta_2^3 = 0,$$

and in a similar way, $-b_0 b_1 b_3^2 + 3b_0 b_1 b_3^2 - b_0 b_2^2 b_3 + b_3 b_0^3 \Sigma\beta_1^2\beta_2^2 = 0$. These two give $b_0^3 \Sigma\beta_1^2\beta_2^2$ and $b_0^2 \Sigma\beta_1^2\beta_2^2$. The third equation gives

$$b_0 b_0^3 \Sigma\beta_1^3\beta_2\beta_3 + b_1 b_0^3 \Sigma\beta_1^2\beta_2\beta_3 + 3b_2 b_0^3 \beta_1\beta_2\beta_3 + b_3 b_0^3 \Sigma\beta_1\beta_2 = 0, \text{ or}$$

$$-b_0 b_3 b_0^2 \Sigma\beta_1^2 + b_0 b_1^2 b_3 - 3b_0^2 b_2 b_3 + b_0^2 b_2 b_3 = 0.$$

It gives $\Sigma\beta_1^2$, but that is known if the previous resultant is calculated. The next equation gives $b_0^4 \Sigma\beta_1^3 + b_1 b_0^3 \Sigma\beta_1^2 - b_0^2 b_1 b_2 + 3b_0^3 b_3 = 0$.

With the preceding it gives $b_0^3 \Sigma\beta_1^3$.

[To be continued.]